

Proton Radius, Darwin–Foldy Term and Radiative Corrections

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Abstract. We discuss the role of the so-called Darwin-Foldy term in the evaluation of the proton and deuteron charge radii from atomic hydrogen spectroscopy and nuclear scattering data. The question of whether this term should be included or excluded from the nuclear radius has been controversially discussed in the literature. We attempt to clarify which literature values correspond to which conventions. A detailed discussion of the conventions appears useful because a recent experiment [R. Pohl *et al.*, Nature **466**, 213 (2010)] has indicated that there is a discrepancy between the proton charge radii inferred from ordinary (“electronic”) atomic hydrogen and muonic hydrogen. We also investigate the role of quantum electrodynamic radiative corrections in the determination of nuclear radii from scattering data, and propose a definition of the nuclear self energy which is compatible with the subtraction of the radiative corrections in scattering experiments.

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1 Introduction

Quite surprisingly, the recent muonic hydrogen Lamb shift experiment [1] at PSI has led to a value of the proton charge radius which is in disagreement with both the 2006 CODATA value of the mean-square proton charge radius [2], as well as in disagreement with the mean-square charge radius derived from electron-proton scattering experiments [3–5]. This disagreement raises a number of questions, two of which are the following.

(i) Is it possible that different conventions have been used in order to infer the mean-square charge radius of the proton in atomic and nuclear physics? In particular, in Ref. [6], the authors advocate to add the so-called Darwin–Foldy correction to the proton charge radius. Yet, its inclusion either into the nuclear radius [6] or into the electron binding energy [7] has been the subject of discussions and is known to depend on the spin of the nucleus [7]. The question is whether the inclusion or exclusion of the Darwin–Foldy correction has been implemented consistently in all determinations of the proton charge radius in atomic and nuclear physics.

(ii) The quantum electrodynamic (QED) radiative corrections to electron-proton scattering have been discussed in a number of papers, notably, Refs. [8–12]. The QED corrections are subtracted before the form factors of the proton are deduced from experiment. The notion is that the electric and magnetic Sachs form factors of the proton (G_E and G_M) should be defined so that they correspond to the internal structure of the proton. The same applies to the mean-square proton charge radius, which is proportional to the slope of the G_E form factor at zero momentum transfer. The corresponding correction in atomic

physics is the so-called nuclear self-energy [13]. The question is whether the subtraction of the radiative corrections in scattering experiments are compatible with the common definition of the nuclear self-energy used in the atomic physics literature.

Here, we attempt to answer both of these questions, and we also investigate the role of QED radiative corrections in the determination of the nuclear radius. Strictly speaking, the slope of the Dirac form factor F_1 and of the Sachs form factor G_E of the proton is known to be infrared divergent for any spin of the nucleus [14], unless quantum electrodynamic (QED) radiative corrections are subtracted. From the atomic physics point of view, this divergence is manifest in a logarithmic term in the nuclear self-energy which contributes to the atomic binding energy. Radiative corrections and infrared bremsstrahlung effects (which depend on the acceptance range of the detectors) are subtracted before evaluating the slope of the form factors. As a cultural matter, this aspect is not mentioned in the pertinent literature [3–5].

From scattering experiments, we have a rather old value from Ref. [15] for the root-mean-square proton charge radius $r_p = \sqrt{\langle r^2 \rangle_p}$, which reads $r_p = 0.88(3)$ fm. It was obtained using a dipole fit to the form factor. In Ref. [3], this value has been confirmed and improved to yield $r_p = 0.880(15)$ fm. Then, according to Refs. [4, 5], the proton radius inferred from the world scattering data reads

$$r_p = 0.895(18) \text{ fm}, \quad (1)$$

in good agreement with the 2008 CODATA value of

$$r_p = 0.8768(69) \text{ fm}. \quad (2)$$

The latter value is mainly inferred from the analysis of spectroscopic data from atomic hydrogen and deuterium spectroscopy [2, 16]. The most recent and accurate measurement of the proton radius from electron scattering [17], yields a value of

$$r_p = 0.879(8) \text{ fm}, \quad (3)$$

when the statistical and systematic uncertainties given in Ref. [17] are added quadratically, in excellent agreement with the CODATA value (2) inferred mainly from atomic spectroscopy and the value (1) inferred from the world average of scattering data. However, there is a large discrepancy with the recent value from the PSI muonic hydrogen experiment, which reads

$$r_p = 0.84184(67) \text{ fm}. \quad (4)$$

Because of this discrepancy, a study of the conventions used in the determination of the proton radius from scattering data and spectroscopy is indicated.

We proceed as follows. First, the role of the Darwin–Foldy correction in atomic and nuclear physics determinations of the proton charge radius is analyzed (Sec. 2). Radiative corrections to the proton line are discussed in Sec. 3. Conclusions are reserved for Sec. 4. SI (MKSA) units are used throughout the article unless stated otherwise.

2 Darwin–Foldy Correction

In Ref. [7], it has been shown that the zitterbewegung term of the nucleus is absent in the atomic binding energy for spin-0 and spin-1 nuclei such as the deuteron. We here use the conventions of Ref. [7], acknowledging that others exist [18]. From the nuclear physics side [6], it has been recommended to include the so-called Darwin–Foldy correction in the value of the proton radius. There is a connection between these two statements which will be explored in the following.

The zitterbewegung term of the nucleus is part of the so-called Barker–Glover corrections to atomic energy levels [19]. The Barker–Glover corrections follow from the two-body Breit Hamiltonian (Chap. 83 of Ref. [20]) and are listed in the last term on the right-hand side of Eq. (10) of Ref. [2]. They read

$$E_{\text{BG}} = \frac{(Z\alpha)^4 m_r^3 c^2}{2n^3 m_N^2} \left(\frac{1}{j+1/2} - \frac{1}{\ell+1/2} \right) (1 - \delta_{\ell 0}), \quad (5)$$

where Z is the nuclear charge number, α is the fine-structure constant, m_r is the reduced mass of the system, m_N is the nuclear mass, c is the speed of light, n the main quantum number, and j and ℓ are the total and the orbital angular momentum quantum numbers of the reference state. Indeed, the last term in this expression is the

Darwin–Foldy term,

$$\begin{aligned} E_{\text{DF}} &= - \frac{(Z\alpha)^4 m_r^3 c^2}{2n^3 m_N^2} \left(\frac{1}{j+1/2} - \frac{1}{\ell+1/2} \right) \delta_{\ell 0} \\ &= \frac{(Z\alpha)^4 m_r^3 c^2}{2n^3 m_N^2} \delta_{\ell 0}, \end{aligned} \quad (6)$$

which is due to the zitterbewegung term of the nucleus. Alternatively, it can be written as

$$E_{\text{DF}} = \frac{2}{3} \left(\frac{m_r}{m_e} \right)^3 \frac{(Z\alpha)^4 m_e c^2}{n^3 \lambda_C^2} \left\{ \frac{3\hbar^2}{4m_N^2 c^2} \right\} \delta_{\ell 0}. \quad (7)$$

Here, α is the fine-structure constant, and Z is the nuclear charge number, m_r is the reduced mass of the system, m_e is the electron mass, c is the speed of light, and λ_C is the Compton wavelength of the electron divided by a factor 2π . The Darwin–Foldy term is nonvanishing only for S states ($\ell = 0$).

The main nuclear-size shift of atomic energy levels is given by the well-known expression

$$E_{\text{NS}} = \frac{2}{3} \left(\frac{m_r}{m_e} \right)^3 \frac{(Z\alpha)^4 m_e c^2}{n^3 \lambda_C^2} \langle r^2 \rangle_N \delta_{\ell 0}, \quad (8)$$

where $\langle r^2 \rangle_N$ is the mean-square charge radius of the nucleus. The expressions (7) and (8) have the same structure. The nuclear size effect and the Darwin–Foldy correction have the same effect on the spectrum if we alternatively add the Darwin–Foldy correction to the atomic energy levels or if we add the term in curly brackets in Eq. (7) to the mean square nuclear radius. Let us refer to the nuclear radius in “atomic physics” (ATP) conventions. Indeed, this definition has implicitly been proposed in the paper [21], where the nuclear radius difference of proton and deuteron was experimentally measured and theoretically evaluated. The following two replacements are found to be equivalent,

$$E_{\text{NS}} \rightarrow E_{\text{NS}} + E_{\text{DF}} \Leftrightarrow \langle r^2 \rangle_N^{\text{ATP}} \rightarrow \langle r^2 \rangle_N^{\text{ATP}} + \langle r^2 \rangle_N^{\text{DF}}, \quad (9)$$

where

$$\langle r^2 \rangle_N^{\text{DF}} = \frac{3\hbar^2}{4m_N^2 c^2}. \quad (10)$$

Here, N denotes the nucleus ($N = p$ for the proton), and m_N is the nuclear mass. According to Ref. [6], in atomic physics conventions, the proton charge radius is proportional to the slope of a subtracted Sachs form factor G_E ,

$$\langle r^2 \rangle_p^{\text{ATP}} = \langle r^2 \rangle_E^p = 6\hbar^2 \left. \frac{\partial G_E(q^2)}{\partial q^2} \right|_{q^2=0}, \quad (11)$$

where $q^2 = (q^0)^2 - \mathbf{q}^2$ is the momentum transfer. In order to be consistent, it is important to stress that a subtracted form factor G_E has to be used in the evaluation of the slope, because formally, the Sachs form factor G_E has an infinite slope at zero momentum transfer, due to an

infrared divergence at the one-loop level, which is caused by vertex corrections involving virtual interactions with very soft virtual photons. This is illustrated in the discussion following Eq. (15) below. The infrared divergence thus is of quantum electrodynamic origin and not a consequence of the internal structure of the proton, and it only enters at the one-loop level (order α). In the literature, the subtractions are sometimes carried out tacitly, and it is understood that the form factor G_E employed for the proton is due entirely to its internal structure, and all QED effects have been subtracted.

Let us set this problem aside for the moment and continue to study the role of the Darwin–Foldy correction in the definition of the nuclear charge radius. An alternative convention for the proton charge radius is being discussed in Ref. [6]. We refer to this convention as the Friar–Martorell–Sprung (FMS) convention as it has been advocated in Ref. [6]. In FMS conventions [see also Eq. (9)], the mean square proton charge radius includes the Darwin–Foldy correction,

$$\langle r^2 \rangle_p^{\text{FMS}} = \langle r^2 \rangle_p^{\text{ATP}} + \langle r^2 \rangle_p^{\text{DF}}. \quad (12)$$

According to Ref. [6], the alternative mean square charge radius can be written as the slope of a modified Sachs form factor

$$\tilde{G}_E(q^2) = \frac{G_E(q^2)}{\sqrt{1 - q^2/4m_p^2}}, \quad (13a)$$

$$\langle r^2 \rangle_p^{\text{FMS}} = 6\hbar^2 \left. \frac{\partial \tilde{G}_E(q^2)}{\partial q^2} \right|_{q^2=0}. \quad (13b)$$

The authors of Ref. [6] show that this representation is better adapted to the relativistically invariant representation of the Rosenbluth [22] formula.

If one uses the FMS convention (12) for the proton charge radius, then, even for a pointlike nucleus, the nuclear size correction is nonvanishing for the atomic binding energy. Therefore, the ATP definition of the charge radii has been favored in Ref. [23], who have argued in Sec. 7.1.1 of the cited literature reference that in Ref. [6], “it is suggested to include the Darwin–Foldy contribution in the definition of the nuclear charge radius. While one can use any consistent definition of the nuclear charge radius, this particular choice seems to us to be unattractive since in this case even a truly pointlike particle in the sense of quantum field theory (say an electron) would have a finite charge radius even in zero-order approximation.” Here, the authors refer to the zeroth-order approximation as the one without any radiative corrections. In order to ease our mind, we may note that the Darwin–Foldy correction vanishes for an infinitely heavy, point nucleus, i.e., in the limit of $m_N \rightarrow \infty$. In this limit, the ATP and FMS conventions for the charge radius definition are in agreement. Due to the uncertainty principle, one cannot locate a particle and therefore, its charge, to better than its Compton wavelength. The Darwin–Foldy correction is of this magnitude.

There is thus an immediate need to clarify which conventions have actually been used in the determinations of the proton charge radius in Refs. [3–5] and in the CODATA adjustment in Ref. [2]. We observe:

(i) According to the first (unnumbered) equation given on p. 410 of Ref. [5], one may infer that in Refs. [3–5], the form factor G_E (or merely an infrared safe, subtracted variant of it) and not \tilde{G}_E is being used for the determination of the rms radius given in Eq. (1); this observation can be confirmed [24]. This means that the values given in Refs. [3–5] and in particular in Eq. (1) are in agreement with the atomic physics conventions mentioned above. The same applies to the values given in Eq. (3).

(ii) In the latest CODATA adjustment given in Ref. [2], the conventions are consistent with those used here and in Ref. [21], although this is somewhat less obvious. First of all, we reemphasize that, as evident from Eq. (10) of Ref. [2], the atomic binding energy is defined to include the Dirac–Foldy term in the CODATA adjustment. For the deuteron, there is no Darwin–Foldy correction in the atomic binding energy (see [7]). In Ref. [2], the authors still use the Darwin–Foldy correction in the atomic part of the energy (even for deuterium) and later add the Darwin–Foldy correction back on to the deuterium radius. This is consistently done in all CODATA adjustments since the 1998 adjustment (see Ref. [25]), and a pertinent discussion can be found in Appendix A8 near Eq. (A56) of Ref. [25]. A superficial reading of Appendix A8 of Ref. [25] might otherwise suggest that the Darwin–Foldy correction has been inadvertently added to the deuteron (rather than proton!) nuclear charge radius in Ref. [25]; but this addition is compensated by the inclusion of the Darwin–Foldy term even for bound-state deuterium energy levels, where according to Ref. [7], this term should have been excluded. Therefore, both proton and deuteron charge radii given in Ref. [2] correspond to ATP conventions [21] and exclude the Darwin–Foldy term from the nuclear radii for both proton and deuteron.

3 Radiative Corrections

We now turn our attention to the role of radiative corrections to the proton line in both nuclear physics scattering experiments as well as in the determination of atomic binding energy levels. We recall once more that in atomic physics (ATP) conventions, the proton charge radius is defined as the slope of a subtracted electric G_E Sachs form factor, $\langle r^2 \rangle_p^{\text{ATP}} = 6\hbar^2 \partial G_E / \partial q^2|_{q^2=0}$. The electric and magnetic G_E and G_M form factors for the proton (a spin- $\frac{1}{2}$ particle) are related to the Dirac F_1 and Pauli F_2 form factors by the relations

$$G_E(q^2) = F_1(q^2) + \frac{q^2}{4(m_p c)^2} F_2(q^2), \quad (14a)$$

$$G_M(q^2) = F_1(q^2) + F_2(q^2), \quad F_2(0) = \kappa_p, \quad (14b)$$

where $\kappa_p = (g_p - 2)/2 = 1.792\,847\,356(23)$ gives the anomalous magnetic moment of the proton (see Ref. [2]).

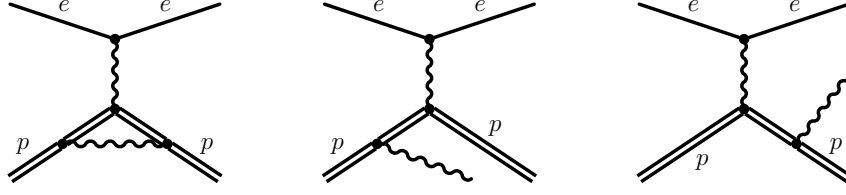


Fig. 1. Radiative vertex correction to electron-proton scattering (heavy line) and associated bremsstrahlung diagrams.

For a point particle like the electron, we have $\kappa_e = \alpha/(2\pi) \ll 1$, but for a particle like the proton, the bulk of the contribution to κ_p is from the internal structure, whereas a tiny correction also is due to its electromagnetic nature.

Let us therefore make the following separation,

$$G_E(q^2) = \overline{G}_E(q^2) + G_E^{\text{QED}}(q^2), \quad (15a)$$

$$F_1(q^2) = \overline{F}_1(q^2) + F_1^{\text{QED}}(q^2), \quad (15b)$$

$$F_2(q^2) = \overline{F}_2(q^2) + F_2^{\text{QED}}(q^2). \quad (15c)$$

$$\kappa_p = \overline{\kappa}_p + \kappa_p^{\text{QED}}. \quad (15d)$$

Here, the overlined quantities represent the contributions to the proton form factor due to its internal structure, whereas the quantities with the superscript QED represent the contributions due to QED point-particle theory.

The one-loop slope of G_E^{QED} is infrared divergent solely due to QED vertex corrections (see Fig. 1), but the slope of \overline{G}_E is infrared safe. Indeed, the slope of the Dirac F_1 form factor of any charged spin- $\frac{1}{2}$ particle is infrared divergent, and the coefficient of the logarithmic infrared divergence of F_1 even is independent of the nuclear spin [14]. The infrared problem persists both in the atomic physics determination of the nuclear charge radius as well as in the determination from scattering data. From Chap. 7 of Ref. [14], we know that the *pure QED* radiative contribution from the first diagram in Fig. 1 corresponds to the following replacement of the Dirac current γ^μ of the proton,

$$\gamma^\mu \rightarrow \gamma^\mu + \left\{ \gamma^\mu \left[F_1^{\text{QED}}(q^2) - 1 \right] + i \frac{\sigma^{\mu\nu} q_\nu}{2m_p c} F_2^{\text{QED}}(q^2) \right\}, \quad (16)$$

where F_1^{QED} and F_2^{QED} are the QED expressions for the Dirac and Pauli form factors of a spin- $\frac{1}{2}$ point particle with the proton mass, as given in Eqs. (7-60) and (7-58) of Ref. [14], respectively. In particular, we have $F_2^{\text{QED}}(0) = \alpha/(2\pi)$. From Eqs. (3.36) and (4.14) of Ref. [9], and from Eq. (A76) of Ref. [10], we may conclude that the radiative QED term in curly brackets in Eq. (16) is indeed subtracted when the radiative corrections to electron-proton scattering are eliminated from experimental scattering data.

We now have to turn our attention to atomic physics and identify the nuclear self-energy as the correction to atomic energy levels corresponding to the terms in curly

brackets in Eq. (16), which are subtracted from scattering data. To this end, we first make a slight detour and observe that the proton radius definition according to

$$\begin{aligned} \langle r^2 \rangle_p^{\text{ATP}} &= 6\hbar^2 \left. \frac{\partial \overline{G}_E(q^2)}{\partial q^2} \right|_{q^2=0} \\ &= 6\hbar^2 \left. \frac{\partial \overline{F}_1(q^2)}{\partial q^2} \right|_{q^2=0} + \frac{3\hbar^2}{2(m_p c)^2} \overline{\kappa}_p, \end{aligned} \quad (17)$$

entails a term originating from the “internal” contribution $\overline{\kappa}_p$ to the anomalous magnetic moment of the proton. We note that $\kappa_p \approx \overline{\kappa}_p$ because $\kappa_p^{\text{QED}} = \alpha/(2\pi) \ll \kappa_p$ is small. The anomalous magnetic moment of the electron shifts S state energy levels [see Eqs. (42) and (44) of Ref. [23]] by

$$\begin{aligned} E_{\kappa_e} &= \left(\frac{\alpha}{2\pi} \right) \frac{(Z\alpha)^4}{n^3} \left(\frac{m_r}{m_e} \right)^3 m_e c^2 \delta_{\ell 0} \\ &= \kappa_e \frac{(Z\alpha)^4}{n^3} \left(\frac{m_r}{m_e} \right)^3 m_e c^2 \delta_{\ell 0}. \end{aligned} \quad (18)$$

The corresponding proton line contribution reads

$$\begin{aligned} E_{\kappa_p} &= \overline{\kappa}_p \frac{(Z\alpha)^4}{n^3} \left(\frac{m_r}{m_p} \right)^2 m_r c^2 \delta_{\ell 0} \\ &= \frac{2}{3} \left(\frac{m_r}{m_e} \right)^3 \frac{(Z\alpha)^4 m_e c^2}{n^3 \lambda_C^2} \left(\frac{3}{2} \frac{\hbar^2}{(m_p c)^2} \overline{\kappa}_p \right) \delta_{\ell 0}. \end{aligned} \quad (19)$$

The slope of the F_1 form factor of the proton leads to an energy correction,

$$E_{F_1} = \frac{2}{3} \left(\frac{m_r}{m_e} \right)^3 \frac{(Z\alpha)^4 m_e c^2}{n^3 \lambda_C^2} \left(6\hbar^2 \left. \frac{\partial \overline{F}_1(q^2)}{\partial q^2} \right|_{q^2=0} \right) \delta_{\ell 0}. \quad (20)$$

The sum of E_{κ_p} and E_{F_1} is

$$\begin{aligned} E_{\text{NS}} &= \frac{2}{3} \left(\frac{m_r}{m_e} \right)^3 \frac{(Z\alpha)^4 m_e c^2}{n^3 \lambda_C^2} \\ &\quad \times \left(6\hbar^2 \left. \frac{\partial \overline{F}_1(q^2)}{\partial q^2} \right|_{q^2=0} + \frac{3}{2} \frac{\hbar^2}{(m_p c)^2} \overline{\kappa}_p \right) \\ &= \frac{2}{3} \left(\frac{m_r}{m_e} \right)^3 \frac{(Z\alpha)^4 m_e c^2}{n^3 \lambda_C^2} \langle r^2 \rangle_p^{\text{ATP}} \delta_{\ell 0} \\ &= \frac{2}{3} \hbar c \langle r^2 \rangle_p^{\text{ATP}} \left\langle \pi(Z\alpha) \delta^3(r) \right\rangle_{nS} \delta_{\ell 0}, \end{aligned} \quad (21)$$

which corresponds to the well-known expression (8) and clarifies that indeed, the anomalous magnetic term of the proton forms part of the nuclear size correction in the ATP conventions, and that indeed, this convention is in agreement with the mean-square-radius of the proton being defined as the slope of the Sachs G_E form factor. The expectation value of the Dirac δ for S states is

$$\left\langle \pi(Z\alpha)\delta^3(r) \right\rangle_{nS} = \frac{(Z\alpha)^3}{n^3} \left(\frac{m_r c}{\hbar} \right)^3. \quad (22)$$

In order to describe the two-body interaction including the nuclear-size effect, we have to consult the two-body Breit Hamiltonian, and temporarily switch to natural units with $\hbar = c = \epsilon_0 = 1$. The Schrödinger Hamiltonian of the two-body system consisting of an orbiting spin- $\frac{1}{2}$ particle of mass m_e (the electron) and a spin- $\frac{1}{2}$ nucleus of mass m_N and nuclear charge number Z is

$$H_S = \frac{\mathbf{p}^2}{2m_r} - \frac{Z\alpha}{r}, \quad (23)$$

where m_r is the reduced mass of the system. The two-body Breit–Pauli Hamiltonian with anomalous magnetic moment corrections for electron and proton reads

$$\begin{aligned} H = & -\frac{\mathbf{p}^4}{8m_e^3} - \frac{\mathbf{p}^4}{8m_p^3} \\ & + \frac{2Z\alpha}{3} \left(\frac{3}{4m_e^2} + \frac{3}{4m_p^2} + \langle r^2 \rangle_p^{\text{ATP}} \right) \pi \delta^3(r) \\ & - \frac{Z\alpha}{2m_e m_p r} \mathbf{p} \left(\frac{\delta^{ij}}{r} + \frac{r^i r^j}{r^3} \right) \mathbf{p} + \underbrace{(1 + 2\kappa_e) \frac{Z\alpha}{4m_e^2 r^3} \mathbf{L} \cdot \boldsymbol{\sigma}_e}_{\text{fs}} \\ & + \underbrace{(1 + \kappa_e) \frac{Z\alpha}{2m_e m_p r^3} \mathbf{L} \cdot \boldsymbol{\sigma}_e}_{\text{fs}} + \underbrace{(1 + 2\kappa_p) \frac{Z\alpha}{4m_p^2 r^3} \mathbf{L} \cdot \boldsymbol{\sigma}_p}_{\text{hfs}} \\ & + \underbrace{(1 + \kappa_p) \frac{Z\alpha}{2m_e m_p r^3} \mathbf{L} \cdot \boldsymbol{\sigma}_p}_{\text{hfs}} + \underbrace{\frac{(1 + \kappa_e)(1 + \kappa_p)Z\alpha}{4m_e m_p r^3}}_{\text{hfs}} \\ & \times \underbrace{\left\{ \frac{8\pi}{3} \boldsymbol{\sigma}_e \cdot \boldsymbol{\sigma}_p \delta^3(r) + 3 \frac{\boldsymbol{\sigma}_e \cdot \mathbf{r} \boldsymbol{\sigma}_p \cdot \mathbf{r}}{r^5} - \frac{\boldsymbol{\sigma}_e \cdot \boldsymbol{\sigma}_p}{r^3} \right\}}_{\text{hfs}}. \quad (24) \end{aligned}$$

Here, we keep the nuclear charge number Z as a variable so that the expression below can be readily generalized to a spin- $\frac{1}{2}$ nucleus with $Z \neq 1$. Terms labeled with “fs” are relevant for the fine structure, whereas terms labeled by “hfs” correspond to the hyperfine structure. All terms in Eq. (24) proportional to the electron anomaly κ_e contribute to the fine structure and thus, to the Lamb shift of these states. Except for the term incorporated into the nuclear mean-square charge radius $\langle r^2 \rangle_p^{\text{ATP}}$, the terms proportional to the (complete) proton anomalous magnetic moment κ_p listed in Eq. (24) influence only the exchange of a magnetic photon of the nucleus and the orbiting particle, i.e., the hyperfine structure (not the Lamb shift).

Hyperfine structure effects are by definition excluded from the Lamb shift [26].

We now have to carefully define the nuclear self-energy for S states so that the QED contributions to the proton factors, which were excluded from the form factors used in Eq. (19), (20), and (21) for the evaluation of the nuclear-size correction, are included instead into the nuclear self-energy. Furthermore, we notice that the “fs” terms, which are proportional to the spin-orbit coupling $\mathbf{L} \cdot \boldsymbol{\sigma}_e$ of the electron, are defined with the phenomenologically inserted anomalous magnetic moment κ_e of the electron. We also have to define the nuclear self-energy so that the terms due to the $\kappa_p^{\text{QED}} = F_2^{\text{QED}}(0)$ part of proton anomalous moment, which are already included in the phenomenologically inserted full anomalous magnetic moment κ_p in the $\mathbf{L} \cdot \boldsymbol{\sigma}_p$ terms in Eq. (24), are not double counted in the nuclear self-energy.

The F_1 form factor of the *electron* induces the following correction to the Lamb shift,

$$E_e = \frac{\alpha(Z\alpha)^4}{\pi n^3} \left(\frac{m_r}{m_e} \right)^3 m_e c^2 \left\{ \frac{4}{3} \ln \left(\frac{m_e}{2\epsilon} \right) + \frac{11}{18} + \frac{1}{2} \right\} \delta_{\ell 0}, \quad (25)$$

where ϵ is a noncovariant, infrared cutoff parameter [27, 28]. The matching with the covariant photon mass has given rise to some discussion [29] in the early days of quantum electrodynamics. The terms 11/18 and 1/2 are from the nonlogarithmic term of the slope of the $F_1 = F_1^{\text{QED}}$ form factor of the *electron*, and the term 1/2 is due to $\kappa_e = \kappa_e^{\text{QED}}$. From the proton line, we have the following, corresponding contribution due to the QED parts F_1^{QED} and F_2^{QED} of the proton form factors,

$$E_{\text{NSE},1} = \frac{Z(Z\alpha)^5}{\pi n^3} \left(\frac{m_r}{m_p} \right)^2 m_r c^2 \left\{ \frac{4}{3} \ln \left(\frac{m_p}{2\epsilon} \right) + \frac{10}{9} \right\} \delta_{\ell 0}, \quad (26)$$

for S states. The infrared divergence in (26), which for scattering experiments is compensated by bremsstrahlung diagrams, is cut off in atomic physics at the binding energy scale $(Z\alpha)^2 m_r$. The matching low-energy contribution [30] contains the Bethe logarithm $\ln k_0(n, \ell)$,

$$\begin{aligned} E_{\text{NSE},2} = & \frac{Z(Z\alpha)^5}{\pi n^3} \left(\frac{m_r}{m_p} \right)^2 m_r c^2 \\ & \times \left\{ \frac{4}{3} \ln \left(\frac{2\epsilon}{(Z\alpha)^2 m_r} \right) \delta_{\ell 0} - \frac{4}{3} \ln k_0(n, \ell) \right\}. \quad (27) \end{aligned}$$

The sum of (26) and (27) is free from the scale separation parameter ϵ and reads

$$\begin{aligned} E_{\text{NSE}} = & \frac{Z(Z\alpha)^5}{\pi n^3} \left(\frac{m_r}{m_p} \right)^2 m_r c^2 \\ & \times \left\{ \left[\frac{4}{3} \ln \left(\frac{m_p}{m_r (Z\alpha)^2} \right) + \frac{10}{9} \right] \delta_{\ell 0} - \frac{4}{3} \ln k_0(n, \ell) \right\}. \quad (28) \end{aligned}$$

This expression generalizes the nuclear self-energy to states with nonvanishing angular momenta.

A few remarks are in order. First, we observe that the anomalous magnetic moment κ_e , due to the spin-orbit coupling of the electron, leads to the following term in the self-energy for non- S states,

$$E' = \frac{\alpha(Z\alpha)^4}{\pi n^3} \left(\frac{m_r}{m_e}\right)^2 m_e c^2 \left(-\frac{1}{2\kappa(2\ell+1)}\right) (1 - \delta_{\ell 0}), \quad (29)$$

where $\kappa = 2(\ell - j)(j + 1/2)$ is the Dirac quantum number. The analogous term due to the proton line is given in Eq. (153) of Ref. [23],

$$E'' = \frac{Z(Z\alpha)^5}{\pi n^3} \frac{m_r}{m_p} m_r c^2 \left(-\frac{1}{2\kappa(2\ell+1)}\right) (1 - \delta_{\ell 0}). \quad (30)$$

However, this term originates from the spin-orbit coupling of the proton and is already contained in the phenomenologically inserted κ_p in Eq. (24). We therefore exclude this term from the definition of the nuclear self-energy for non- S states.

The Bethe logarithm term $\ln k_0(n, \ell)$ in Eq. (28), for a general hydrogenic state, can be derived on the basis of nonrelativistic perturbation theory for the two-body system [31]. It is a consequence of the low-energy part of the proton self energy, which is due to the exchange of low-energy photons along the proton line. The logarithmic term (in ϵ) in the low-energy part given in Eq. (27) compensates the infrared divergence of the F_1^{QED} form factors at the atomic binding energy scale, in analogy to the cancellation of infrared divergences in the vertex and bremsstrahlung diagrams (for scattering). The definition (28) takes into account the quantum electrodynamic properties of the proton (as a charged spin- $\frac{1}{2}$ particle) and is compatible with the radiative corrections listed in Eq. (16). The nonlogarithmic term $10/9$ in Eq. (28), which constitutes an addition to expression for the nuclear self-energy given in Ref. [13], corresponds to a shift of the proton radius by about $\delta r_p = 0.0001$ fm or alternatively to 0.20 kHz in frequency units for the hydrogen-deuterium isotope shift of the $1S$ – $2S$ transition.

Finally, one might argue with respect to our result (28) that the starting point of QED calculations connected with the proton is a proton with form factors that are due to strong interactions, and that, therefore, all QED integrals are cut off from above by the proton radius, i.e., that the proton structure significantly influences the exchange of virtual photons of wavelengths shorter than its own size. However, the ϵ parameter is an infrared cutoff, and it stems from the infrared divergence of the F_1^{QED} form factor of the proton. Indeed, the physically appropriate values for the ϵ parameter are in the region

$$(Z\alpha)^2 m_e \ll \epsilon \ll m_e \ll \Lambda, m_p, \quad (31)$$

where $\Lambda = \sqrt{0.71 \text{ GeV}^2} = 0.842 \text{ GeV}$ is a parameter characterizing the proton size. The first inequality in Eq. (31) is due to the requirement that all bound-state poles must be integrated over in evaluating the low-energy part of the self-energy, and the latter inequality is due to the

relativistic nature of the electron mass scale (separation from the high-energy part). The coefficient multiplying the logarithmic infrared divergence in Eq. (26) is the same whether we use $\ln[\Lambda/(2\epsilon)]$ for the logarithmic term, as done in Eq. (154) of Ref. [23], or $\ln[m_p/(2\epsilon)]$, with $m_p = 0.938 \text{ GeV}$, as done here. The proton structure thus does not influence the leading coefficient of the logarithmic divergence of the infrared behavior of the form factor. The question then is which portion of the proton structure should be counted as a radiative nuclear self energy, and which should be counted as a proton structure correction. Here, we advocate the view that if the radiative corrections to scattering are accounted for by the F_1^{QED} and F_2^{QED} contributions to the form factors of the proton (treated as a point-like QED particle), then, also, the proton should be treated as a point-like QED particle in evaluating the nuclear self energy in atomic physics. The difference of the above mentioned two logarithms,

$$\ln\left(\frac{\Lambda}{2\epsilon}\right) - \ln\left(\frac{m_p}{2\epsilon}\right) = \ln\left(\frac{\Lambda}{m_p}\right), \quad (32)$$

is infrared safe, part of \overline{F}_1 and therefore, part of the nuclear-size correction in our conventions. While this view is at variance with the result obtained in Eq. (154) Ref. [23], we stress that the precise definition of the nuclear self energy depends on the way in which radiative corrections to the proton line are subtracted in atomic physics and in scattering experiments. Our view thus holds relative to the conventions used for subtracting radiative corrections to the proton line in scattering experiments.

In the evaluation of higher-order nuclear structure corrections to the bound-state spectrum, we only have to remember that the G_E inferred from scattering experiments is not the full G_E of the proton, but with the contributions from F_1^{QED} and F_2^{QED} subtracted. The QED contributions to the proton form factors thus have to be treated separately. One example can immediately be given as follows. It is known [32] that the third Zemach moment correction [33] to bound-state energy levels is obtained in second order perturbation theory involving the correction to the Coulomb potential due to the finite size of the nucleus, or in other words, in the second order with respect to the slope of the proton form factor. The contribution from the radiative correction due to the QED contribution $F_1^{\text{QED}}(q^2)$ to the Dirac form factor of the proton is thus neglected in the derivation of the Zemach correction. If desired, this effect can be added back on. It is easy to write down the leading logarithm

$$E^{(2)}(nS) = -\frac{8}{27} \frac{\alpha^2(Z\alpha)^6}{\pi^2 n^3} \left(\frac{m_r}{m_p}\right)^2 m_r c^2 \times \ln^2\left(\frac{m_p}{m_r(Z\alpha)^2}\right) \ln[(Z\alpha)^{-2}] \quad (33)$$

for the energy correction to S states in the second order of perturbation theory with respect to the proton self-energy. The numerical value of this correction is tiny because it is of high order in the electron-proton mass ratio.

4 Conclusions

In this paper, we have analyzed the role of the Darwin–Foldy term in the determination of atomic energy levels and in nuclear physics scattering experiments that measure the proton form factors. In Sec. 2, we have clarified that all current determinations of the proton charge radius, both in atomic as well as in nuclear physics, are based on the slope of the electric Sachs G_E form factor (11) and thus exclude the Darwin–Foldy term. This applies to both the value given in Eq. (1) as well as the value given in Eq. (2). The latter value is determined on the basis of a comparison of the most accurately measured transition frequencies in atomic hydrogen to theory [16].

Furthermore, we have clarified that although a superficial reading of Appendix A8 of Ref. [25] might suggest that the Darwin–Foldy correction has been inadvertently added to the deuteron (rather than proton!) nuclear charge radius in Ref. [25], this is actually not the case: the Darwin–Foldy correction is absent for deuterium [7], and if one includes this contribution into atomic energy levels, even for deuterium, as done by the authors of Refs. [2, 25], then one has to add the Darwin–Foldy correction back on to the deuteron radius as is done in Appendix A8 of Ref. [25] (as well as, consistently, in all CODATA adjustments since 1998).

These observations imply that the Darwin–Foldy term is excluded from both values of the proton radii given in Eqs. (1) and (2), which in addition are in good mutual numerical agreement. In Ref. [1], the proton radius is given as $r_p = 0.84184(67)$ fm, in disagreement with Eqs. (1) and (2), and with the Darwin–Foldy term likewise being excluded. A conceivable accidental incompatibility of the conventions used in Refs. [1–5] for the proton radius therefore cannot be the reason for the observed discrepancy.

In Sec. 3, we turn our attention to radiative corrections to electron–proton scattering. Indeed, the infrared divergence of the Sachs G_E form factor at the one-loop level enters both the atomic physics determination of the proton radius as well as the nuclear physics determination from, e.g., scattering data. While it is clearly stated in Refs. [3–5] that the proton charge radius is determined based on scattering data for G_E , it is useful to remark that the slope of the G_E form factor is determined in each case from *subtracted* scattering data: namely, the infrared divergent, QED radiative corrections whose exact values depend on the acceptance range of the detectors are subtracted when the slope of the form factor is being calculated. Theoretical input data for the radiative corrections is available from a number of sources, notably, Refs. [8–12]. Equation (28) represents an attempt to define the nuclear self energy in full compatibility with the subtractions of radiative corrections to electron–proton scattering, notably, Eqs. (3.36) and (4.14) of Ref. [9], and Eq. (A76) of Ref. [10]. The result (28) also represents a generalization of the nuclear self energy to non- S states ($\ell \neq 0$).

Finally, let us remember that the nuclear self energy is a numerically rather tiny correction, especially for systems with a small mass ratio of the orbiting particle versus the nucleus. Likewise, the consistent subtraction of the radia-

tive corrections listed in Eq. (16) in a scattering experiment depends on the availability of accurate data at low momentum transfer, and on the accurate subtraction of other contributions, such as double scattering (including effects due to the polarizability of the proton) and double bremsstrahlung. It is known [8–12] that the contribution of the latter effects may be numerically more significant than that of the QED terms in curly brackets in Eq. (16). Currently, the uncertainty induced by the nonlogarithmic terms is negligible on the scale of the total uncertainty of the proton radius, but still, it is an important conceptual issue to carry out the analysis of radiative corrections to the proton line consistently in atomic and nuclear physics experiments.

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